Electromagnetic vector potentials and the scalarization of sources in a nonhomogeneous medium

Natalia K. Georgieva*
Department of Electrical and Computer Engineering, McMaster University, Hamilton, Ontario, Canada L8S 4K1

Werner S. Weiglhofer†
Department of Mathematics, University of Glasgow, Glasgow G12 8QW, Great Britain

(Received 15 March 2002; revised manuscript received 7 May 2002; published 21 October 2002)

Electromagnetic source equivalence is considered for the case of an isotropic nonhomogeneous medium. Equivalent transformations of the transversally oriented (with respect to a chosen axis) current sources into longitudinally oriented sources are derived. They allow the reduction of any given distribution of arbitrarily oriented sources to an equivalent distribution of single-component parallel electric and magnetic sources. The technique is referred to as source scalarization; and, together with a recently developed vector potential field representation in an isotropic, nonhomogeneous, lossy medium, which may contain sources of arbitrary orientation, it is applied to produce a complete description of the field in terms of two scalar wave potentials. The proposed source scalarization technique is illustrated by a simple numerical example: the radiation of an electromagnetic pulse by an asymmetrical loop of magnetic currents.

DOI: 10.1103/PhysRevE.66.046614 PACS number(s): 41.20.Jb, 03.50.De, 02.70.−c

I. INTRODUCTION

It is well known that there are certain sources, e.g., a small loop of electric current and its equivalent magnetic dipole, which produce the same field. In general, two sets of sources are considered equivalent with respect to a region of interest V if they produce identical fields in it. So far, the studies on electromagnetic source equivalence have been limited mostly to time-harmonic fields in homogeneous media. For example, the equivalence relations between electric current densities on one hand and magnetic current densities on the other have been known for some decades [1]. They are based on the Helmholtz equation for the electric field vector $\mathbf{E}$ or the magnetic field vector $\mathbf{H}$ (it is noted that throughout the paper vectors appear in bold face, and unit vectors such as $\hat{\mathbf{n}}$ are identified by carrying a $\hat{}$) and lead to field equivalence outside the volume of the sources. More recently, it was shown [2,3] that a given electric current source can be decomposed into a part radiating a transverse-electric (TE) wave and a part radiating a transverse-magnetic (TM) wave with respect to a chosen (or distinguished) axis $\hat{\mathbf{n}}$. This decomposition is based on the equivalence derived in Ref. [1] and is also limited to homogeneous media.

Here, we focus on a problem which, to the best of our knowledge, has not been addressed so far. This is the reduction of a given distribution of electric and magnetic current densities of arbitrary orientation to equivalent sources in the form of single-component electric and magnetic current densities parallel to a distinguished axis $\hat{\mathbf{n}}$. The technique is referred to as source scalarization. An important merit of this technique is that it is valid in *nonhomogeneous* isotropic media. It is applicable not only to the direct field analysis in terms of $\mathbf{E}$ and $\mathbf{H}$ but also to the analysis based on vector and scalar potentials. The transformations allowing the scalarization of the electromagnetic sources are derived using a vector-potential representation of the field. The fields due to the original and the equivalent sources are shown to be identical everywhere, the locations of the sources included.

The significance of the problem of source scalarization stems from its relation to the electromagnetic (EM) field scalarization (or TE/TM field decomposition) with respect to a distinguished axis $\hat{\mathbf{n}}$ in linear media. The purpose of the field scalarization is to represent the field in terms of two scalar potential functions, e.g., a pair of $\hat{\mathbf{n}}$-oriented magnetic and electric vector potentials $\mathbf{A}=\hat{\mathbf{n}}A_n$ and $\mathbf{F}=\hat{\mathbf{n}}F_n$ [4]. A similar technique uses a pair of $\hat{\mathbf{n}}$–oriented magnetic and electric Hertz potentials [5]. These scalar functions, e.g., A_n and F_n, satisfy the wave equation in the time domain or the Helmholtz equation in the frequency domain and are, therefore, called scalar wave potentials. The A_n potential represents a TM_n field and the F_n potential represents a TE_n field. The total field is thus given as the superposition of the TM_n and TE_n fields, which are commonly referred to as field modes.

In a homogeneous region, the direction of the magnetic and electric vector potentials is that of the electric and magnetic current densities, respectively. Thus, if all sources are parallel to $\hat{\mathbf{n}}$, a complete solution in terms of two *decoupled* scalar wave potentials is possible [4–6]. It can be shown that problems involving media whose constitutive parameters are functions of one coordinate only, e.g., along $\hat{\mathbf{n}}$, can also be reduced to the analysis of two decoupled modes, TM_n and TE_n [7,8], if the sources are parallel to $\hat{\mathbf{n}}$. Using the proposed source scalarization technique, the sources in the above EM problems can be transformed to achieve $\hat{\mathbf{n}}$ orientation. Subsequently, the analysis can be carried out either for the TE_n mode or for the TM_n mode. The advantages of solving a single scalar wave equation as opposed to the Maxwell equations for the field vectors are obvious.

The proposed source scalarization technique is in general

*Electronic address: talia@mcmaster.ca
†Electronic address: wsw@maths.gla.ac.uk

valid in a nonhomogeneous medium where the gradients of the constitutive parameters are not restricted to any particular axis. This makes it applicable to problems of a more general type than those described above where a solution can be constructed using either the Maxwell equations or suitable scalar wave potential techniques [9,10]. In the latter, the solution is obtained in terms of two *coupled* scalar wave potentials.

The field and source scalarization in terms of scalar wave potentials is derived and demonstrated in the time-domain making the assumption of an instantaneous response of the medium. However, all equations are in principle directly transferable into the frequency domain, where proper care can be taken of the frequency-dependent constitutive parameters; such a frequency-domain approach became available recently [11].

II. VECTOR POTENTIALS AND SOURCES IN A NONHOMOGENEOUS MEDIUM

A. General vector potential equations

Here, we propose a vector potential formalism for the case of an isotropic nonhomogeneous lossy medium, which may involve electric and magnetic currents of arbitrary directions. It gives a more general definition of the sources of the vector potentials and serves as a starting point toward the scalarization of EM sources. It also gives explicitly the necessary conditions for the EM field scalarization and shows how the coupling between the vector potential components occurs in a nonhomogeneous medium.

For a linear EM problem, one can use the superposition principle and decompose the EM field accordingly. The part for which

$$\nabla\cdot\mathbf{B}^A=\mathbf{0} \tag{1}$$

holds is called the A field, $(\mathbf{E}^A,\mathbf{H}^A)$. The F field, $(\mathbf{E}^F,\mathbf{H}^F)$, is that which fulfills

$$\nabla\cdot\mathbf{D}^F=\mathbf{0}. \tag{2}$$

In general, the constitutive relations for time-dependent fields are given in terms of integrals of convolution type to fulfill the requirement of causality [see, for example, Ref. [12]]. In the following analysis, however, we use the time-dependent constitutive relations of an isotropic, nonhomogeneous medium in their instantaneous form. We have

$$\begin{aligned}\mathbf{D}(\mathbf{x},t)&=\epsilon(\mathbf{x})\mathbf{E}(\mathbf{x},t),\\ \mathbf{B}(\mathbf{x},t)&=\mu(\mathbf{x})\mathbf{H}(\mathbf{x},t),\\ \mathbf{J}_{\sigma_e}(\mathbf{x},t)&=\sigma_e(\mathbf{x})\mathbf{E}(\mathbf{x},t),\\ \mathbf{J}_{\sigma_m}(\mathbf{x},t)&=\sigma_m(\mathbf{x})\mathbf{H}(\mathbf{x},t),\end{aligned} \tag{3}$$

where ϵ is the dielectric permittivity, μ is the magnetic permeability, σ_e is the specific electric conductivity, σ_m is the specific magnetic conductivity of the medium, all depending on the position vector $\mathbf{x}=(x,y,z)$. It is briefly noted that the magnetic conductivity, while a fictitious property of matter, is of considerable value in computational electrodynamics when dispersion-free media are simulated and fictitious perfectly matched absorbers are constructed [13]. In the following derivations, it will be understood that the field and its associated potentials are functions of space and time, while the constitutive parameters are functions of space only. The local nature of the relations in Eq. (3) means that the medium has instantaneous, and thus nonphysical, response. Such an assumption is often made in transient computational algorithms for media whose constitutive parameters are nearly dispersion-free in the frequency band of interest.

In view of Eqs. (1)–(3), the magnetic vector potential $\mathbf{A}$ and the electric vector potential $\mathbf{F}$ are introduced as

$$\begin{aligned}\mu\mathbf{H}^A&=\nabla\times\mathbf{A},\\ \epsilon\mathbf{E}^F&=-\nabla\times\mathbf{F}.\end{aligned} \tag{4}$$

Equations (1) and (2) require that the respective charge densities vanish, $\rho_m^A\equiv0$, $\rho_e^F\equiv0$. We add for clarity that while the magnetic charges and magnetic currents are considered fictitious EM sources, they play an important role in solutions based on the equivalence principle and EM duality [14]. According to the continuity relations, the following restrictions are imposed on the current sources associated with the A field and the F field:

$$\begin{aligned}\nabla\cdot\mathbf{J}_m^A&=\mathbf{0},\\ \nabla\cdot\mathbf{J}_e^F&=\mathbf{0},\end{aligned} \tag{5}$$

where $\mathbf{J}_m^A$ is the magnetic current density which appears as a source of the A field, and $\mathbf{J}_e^F$ denotes the electric current density associated with the F field. Then $\rho_m^A\equiv0$ and $\rho_e^F\equiv0$ hold indeed if zero initial conditions are assumed for the charge distributions: $\rho_{m(t=0)}^A=0$, $\rho_{e(t=0)}^F=0$. Thus, $\mathbf{J}_m^A$ and $\mathbf{J}_e^F$, if represented as the curls of given vector fields, are admissible sources of $\mathbf{A}$ and $\mathbf{F}$, respectively. The current sources of an EM problem can thus be represented as

$$\begin{aligned}\mathbf{J}_e&=\mathbf{J}_e^A+\mathbf{J}_e^F,\\ \mathbf{J}_m&=\mathbf{J}_m^A+\mathbf{J}_m^F,\end{aligned} \tag{6}$$

where the electric current sources of the A field, $\mathbf{J}_e^A$, and the magnetic current sources of the F field, $\mathbf{J}_m^F$, are not subjected to any conditions, while $\mathbf{J}_m^A$ and $\mathbf{J}_e^F$ are a subject to the conditions given in Eq. (5). Here, it should be noted that the classical interpretation is that the sources of $\mathbf{A}$ can be only electrical currents (i.e., $\mathbf{J}_m^A=\mathbf{0}$), and the sources of $\mathbf{F}$ can be only magnetic currents (i.e., $\mathbf{J}_e^F=\mathbf{0}$), which is only a special case of Eq. (5).

The above discussion permits a generalization of the vector potential sources. In addition to $\mathbf{J}_e^A$ and $\mathbf{J}_m^F$, a set of sources $(\mathbf{J}_e^s,\ \mathbf{J}_m^s)$ is introduced via Helmholtz' theorem

$$\begin{aligned}\mathbf{J}_e^s&=\nabla\times\boldsymbol{\mathcal{H}}-\nabla P_e,\\ \mathbf{J}_m^s&=-\nabla\times\boldsymbol{\mathcal{E}}-\nabla P_m,\end{aligned} \tag{7}$$

where $\boldsymbol{\mathcal{H}}$, $\boldsymbol{\mathcal{E}}$, P_e and P_m are given functions of space and time. The current densities $\mathbf{J}_e^s$ and $\mathbf{J}_m^s$ have their divergence-free part clearly distinguished from their curl-free part. The vectors $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{E}}$ can be interpreted as an incident magnetic and electric field, respectively, while the scalar functions P_e and P_m are representative of conservative fields. We refer to the functions $\boldsymbol{\mathcal{H}}$, $\boldsymbol{\mathcal{E}}$, P_m, and P_e as *primary sources*, while $\mathbf{J}_e^s$ and $\mathbf{J}_m^s$ are called the *secondary sources*. The currents $\mathbf{J}_e^A$ and $\mathbf{J}_m^F$, which are not given in terms of a solenoidal and a gradient part, will be distinguished from the secondary sources by referring to them simply as *current sources*.

The divergence-free secondary sources

$$\mathbf{J}_e^{sF} = \nabla \times \boldsymbol{\mathcal{H}},$$
$$\mathbf{J}_m^{sA} = -\nabla \times \boldsymbol{\mathcal{E}} \tag{8}$$

are introduced as admissible sources of **F** and **A**, respectively, according to Eq. (5). The curl-free secondary sources

$$\mathbf{J}_e^{sA} = -\nabla P_e,$$
$$\mathbf{J}_m^{sF} = -\nabla P_m \tag{9}$$

can only be associated with the A field and with the F field, respectively. Thus, the A field and the F field have their sources generalized to include the primary sources in the form

$$\mathbf{J}_e^A + \mathbf{J}_e^{sA} = \mathbf{J}_e^A - \nabla P_e, \quad \mathbf{J}_m^{sA} = -\nabla \times \boldsymbol{\mathcal{E}}, \quad A \text{ field},$$
$$\mathbf{J}_m^F + \mathbf{J}_m^{sF} = \mathbf{J}_m^F - \nabla P_m, \quad \mathbf{J}_e^{sF} = \nabla \times \boldsymbol{\mathcal{H}}, \quad F \text{ field}. \tag{10}$$

Both source sets, the current sources ($\mathbf{J}_e^A$, $\mathbf{J}_m^F$) and the secondary sources ($\mathbf{J}_e^s$, $\mathbf{J}_m^s$), are included in the time-domain Maxwell equations

$$\mathcal{T}_\epsilon \mathbf{E} = \nabla \times \mathbf{H} - \mathbf{J}_e^s - \mathbf{J}_e^A,$$
$$\mathcal{T}_\mu \mathbf{H} = -\nabla \times \mathbf{E} - \mathbf{J}_m^s - \mathbf{J}_m^F. \tag{11}$$

Here, the linear scalar differential operators in time $\mathcal{T}_\epsilon$ and $\mathcal{T}_\mu$ are given by

$$\mathcal{T}_\epsilon = \epsilon \partial_t + \sigma_e,$$
$$\mathcal{T}_\mu = \mu \partial_t + \sigma_m. \tag{12}$$

The operators $\mathcal{T}_\epsilon$ and $\mathcal{T}_\mu$, convenient for use in the time domain, also allow the direct transfer of the time-domain analysis presented below into the frequency domain by replacing $\mathcal{T}_\epsilon$ with $j\omega\tilde{\epsilon}$, and $\mathcal{T}_\mu$ with $j\omega\tilde{\mu}$. Therein, $\tilde{\epsilon}$ is the complex dielectric permittivity and $\tilde{\mu}$ is the complex magnetic permeability.

As a result of the substitution of Eq. (4) into Eq. (11), the following field-to-potential representations are obtained

$$\mathbf{E} = -\mathcal{T}_\mu \mathbf{A}_\mu - \nabla \Phi - \nabla \times \mathbf{F}_\epsilon + \boldsymbol{\mathcal{E}},$$
$$\mathbf{H} = -\mathcal{T}_\epsilon \mathbf{F}_\epsilon - \nabla \Psi + \nabla \times \mathbf{A}_\mu + \boldsymbol{\mathcal{H}}; \tag{13}$$

and, equivalently,

$$\mathcal{T}_\epsilon \mathbf{E} = \nabla \times (\nabla \times \mathbf{A}_\mu - \mathcal{T}_\epsilon \mathbf{F}_\epsilon) - \mathbf{J}_e^A + \nabla P_e,$$
$$\mathcal{T}_\mu \mathbf{H} = \nabla \times (\nabla \times \mathbf{F}_\epsilon + \mathcal{T}_\mu \mathbf{A}_\mu) - \mathbf{J}_m^F + \nabla P_m. \tag{14}$$

During the derivations, the modified vector potentials

$$\mathbf{A}_\mu = \mu^{-1} \mathbf{A},$$
$$\mathbf{F}_\epsilon = \epsilon^{-1} \mathbf{F} \tag{15}$$

have been introduced. From the equivalence of Eqs. (13) and (14), the generalized Lorenz gauge is derived as

$$-\mathcal{T}_\epsilon \Phi = \nabla \cdot \mathbf{A}_\mu + P_e,$$
$$-\mathcal{T}_\mu \Psi = \nabla \cdot \mathbf{F}_\epsilon + P_m. \tag{16}$$

The governing equations of the modified vector potentials are, henceforth, obtained as

$$\nabla^2 \mathbf{A}_\mu - \mathcal{T}_{\mu\epsilon} \mathbf{A}_\mu + (\nabla \mathcal{T}_\epsilon) \times \mathbf{F}_\epsilon - (\nabla \mathcal{T}_\epsilon) \mathcal{T}_\epsilon^{-1} (\nabla \cdot \mathbf{A}_\mu + P_e)$$
$$= -\mathbf{J}_e^A - \mathcal{T}_\epsilon \ \boldsymbol{\mathcal{E}},$$
$$\nabla^2 \mathbf{F}_\epsilon - \mathcal{T}_{\mu\epsilon} \mathbf{F}_\epsilon - (\nabla \mathcal{T}_\mu) \times \mathbf{A}_\mu - (\nabla \mathcal{T}_\mu) \mathcal{T}_\mu^{-1} (\nabla \cdot \mathbf{F}_\epsilon + P_m)$$
$$= -\mathbf{J}_m^F - \mathcal{T}_\mu \boldsymbol{\mathcal{H}}, \tag{17}$$

where $\mathcal{T}_\epsilon^{-1}$ and $\mathcal{T}_\mu^{-1}$ are the inverse of the operators defined in Eq. (12). In Eq. (17), $\mathcal{T}_{\mu\epsilon}$ is a second-order differential operator in time given by

$$\mathcal{T}_{\mu\epsilon} = \mathcal{T}_\mu \mathcal{T}_\epsilon = \mu\epsilon \partial_{tt} + (\epsilon\sigma_m + \mu\sigma_e)\partial_t + \sigma_e \sigma_m. \tag{18}$$

The vector operators $(\nabla \mathcal{T}_\epsilon)$ and $(\nabla \mathcal{T}_\mu)$ are the gradients of the operators defined in Eq. (12):

$$(\nabla \mathcal{T}_\epsilon) = (\nabla \epsilon)\partial_t + (\nabla \sigma_e),$$
$$(\nabla \mathcal{T}_\mu) = (\nabla \mu)\partial_t + (\nabla \sigma_m), \tag{19}$$

so that, for example,

$$(\nabla \mathcal{T}_\epsilon)\Phi = (\nabla \epsilon)\partial_t \Phi + (\nabla \sigma_e)\Phi,$$
$$(\nabla \mathcal{T}_\epsilon) \times \mathbf{F}_\epsilon = (\nabla \epsilon) \times \partial_t \mathbf{F}_\epsilon + (\nabla \sigma_e) \times \mathbf{F}_\epsilon. \tag{20}$$

These vector operators reflect the influence of the material nonhomogeneities. In a homogeneous medium, where $(\nabla \mathcal{T}_\epsilon) = (\nabla \mathcal{T}_\mu) = \mathbf{0}$, Eq. (17) simplifies to two decoupled equations for $\mathbf{A}_\mu$ and $\mathbf{F}_\epsilon$, as per

$$\mathcal{D}\mathbf{A}_\mu = -\mathbf{J}_e^A - \mathcal{T}_\epsilon \ \boldsymbol{\mathcal{E}},$$
$$\mathcal{D}\mathbf{F}_\epsilon = -\mathbf{J}_m^F - \mathcal{T}_\mu \ \boldsymbol{\mathcal{H}}. \tag{21}$$

Therein, $\mathcal{D} = \nabla^2 - \mathcal{T}_{\mu\epsilon}$ is a wave operator that generalizes the *d'Alembert* operator $(\nabla^2 - \mu\epsilon\partial_{tt})$. In view of Eqs. (17) and (21), the potential sources are now defined as

$$\mathbf{G}^A = \mathbf{J}_e^A + \mathcal{T}_\epsilon \boldsymbol{\mathcal{E}},$$

$$\mathbf{G}^F = \mathbf{J}_m^F + \mathcal{T}_\mu \boldsymbol{\mathcal{H}}. \tag{22}$$

It is important to note that the scalar primary sources P_e and P_m affect neither the left-hand side nor the sources in Eq. (21) for the modified vector potentials $\mathbf{A}_\mu$ and $\mathbf{F}_\epsilon$ in a homogeneous medium. This means that the currents in Eq. (9) are nonradiating sources in the sense that they do not affect the EM field outside their own volume. However, they do affect the field vectors locally as seen from Eq. (14), as well as from Eqs. (13) and (16). This behavior conforms with the theory of nonradiating sources in a homogeneous medium [15]. If the medium is not homogeneous ($\boldsymbol{\nabla}\mathcal{T}_\epsilon \neq \mathbf{0}$ and/or $\boldsymbol{\nabla}\mathcal{T}_\mu \neq \mathbf{0}$), P_e, and P_m affect the left-hand sides of Eq. (17). In such a case, the secondary sources (9) can not be classified as nonradiating.

The formalism based on Eq. (17) involves both vector potentials $\mathbf{A}_\mu$ and $\mathbf{F}_\epsilon$ with their six components. It is a more general representation of the EM field in terms of vector potentials than the commonly used magnetic vector potential $\mathbf{A}$. Unlike the latter, the formalism can handle (generally) nonhomogeneous, lossy, isotropic dielectric-magnetic media which involve both electric and magnetic current densities of arbitrary orientations, and this constitutes a different result. The solution procedure involves two coupled second-order vector equations and its computational advantages will become apparent after a scalarization (reduction to two scalar equations) has been carried out.

B. Scalarization of the vector potential equations

Let us assume that the constitutive parameters depend on a single variable along the axis $\hat{\mathbf{n}}$ in the local coordinate system $(\hat{\mathbf{n}}, \hat{\boldsymbol{\tau}}_1, \hat{\boldsymbol{\tau}}_2)$, where $\hat{\mathbf{n}} = \hat{\boldsymbol{\tau}}_1 \times \hat{\boldsymbol{\tau}}_2$. Then both vector operators $(\boldsymbol{\nabla}\mathcal{T}_\epsilon)$ and $(\boldsymbol{\nabla}\mathcal{T}_\mu)$ are parallel to $\hat{\mathbf{n}}$. This corresponds to a type of nonhomogeneity often encountered in practical problems: a locally flat interface between two materials. From Eq. (17) it is obvious that in this case single-component vector potentials defined as $\mathbf{A} = A_{\mu n}\hat{\mathbf{n}}$ and $\mathbf{F} = F_{\epsilon n}\hat{\mathbf{n}}$ are not mutually coupled as the cross products in the left-hand side vanish. Moreover, they do not give rise to vector potential components tangential to the material interface since the vector operators $(\boldsymbol{\nabla}\mathcal{T}_\epsilon)$ and $(\boldsymbol{\nabla}\mathcal{T}_\mu)$ have only $\hat{\mathbf{n}}$ components. They are sufficient by themselves to provide a complete field description.

The orientation of the source terms determines the orientation of the vector potentials as seen from Eq. (17). The orientation of the vector potentials is also important when their boundary conditions must be satisfied at conducting edges. The boundary conditions for the vector potential components, which are tangential to a perfectly conducting edge (the term perfect conductor referring to both perfect electric conductor and perfect magnetic conductor), are well posed, while the boundary conditions for the vector potential components, which are transversal to the edge, are ill posed [9].

In summary, an EM problem can be described by a pair of decoupled collinear vector potentials $(A_{\mu n}, F_{\epsilon n})\hat{\mathbf{n}}$ in regions where: (i) all sources ($\mathbf{J}_e^A$, $\mathbf{J}_m^F$, $\boldsymbol{\mathcal{E}}$, and $\boldsymbol{\mathcal{H}}$) are parallel to $\hat{\mathbf{n}}$; (ii) the gradients of the constitutive parameters are parallel to $\hat{\mathbf{n}}$; and (iii) any perfectly conducting edges are parallel to $\hat{\mathbf{n}}$.

Such regions are said to have a distinguished axis $\hat{\mathbf{n}}$. The source scalarization technique proposed here allows the reduction of electric and magnetic currents of any distribution and orientation to the form $\mathbf{J}_e^A = J_{en}^A\hat{\mathbf{n}}$, $\mathbf{J}_m^F = J_{mn}^F\hat{\mathbf{n}}$, $\boldsymbol{\mathcal{E}} = \mathcal{E}_n\hat{\mathbf{n}}$, and $\boldsymbol{\mathcal{H}} = \mathcal{H}_n\hat{\mathbf{n}}$, thus ensuring condition (i). If all of the above conditions are observed, the two coupled vector equations in Eq. (17) reduce to two decoupled scalar equations for the wave potentials $A_{\mu n}$ and $F_{\epsilon n}$:

$$\mathcal{D}A_{\mu n} - (\partial_n\mathcal{T}_\epsilon)\mathcal{T}_\epsilon^{-1}(\partial_n A_{\mu n} + P_e) = -J_{en}^A - \mathcal{T}_\epsilon\mathcal{E}_n,$$

$$\mathcal{D}F_{\epsilon n} - (\partial_n\mathcal{T}_\mu)\mathcal{T}_\mu^{-1}(\partial_n F_{\epsilon n} + P_m) = -J_{mn}^F - \mathcal{T}_\mu\mathcal{H}_n, \tag{23}$$

where ∂_n represents the $\hat{\mathbf{n}}$ component of the $\boldsymbol{\nabla}$ operator, $\boldsymbol{\nabla} = \partial_n\hat{\mathbf{n}} + \boldsymbol{\nabla}_\tau$. For example, problems involving homogeneous or stratified media are readily reduced to this form if their sources have been scalarized to satisfy condition (i).

Conditions (ii) and (iii) are related to the influence of *implicit* EM sources induced at material nonhomogeneities and perfectly conducting edges. The implicit sources are different from the explicitly defined EM sources ($\mathbf{J}_e^A$, $\mathbf{J}_m^F$, $\boldsymbol{\mathcal{E}}$, and $\boldsymbol{\mathcal{H}}$) as they are dependent on the field that induces them, i.e., they depend on $A_{\mu n}$ and $F_{\epsilon n}$. For example, the implicit current densities $\mathbf{J}_e^i$ and $\mathbf{J}_m^i$ in a nonhomogeneous medium are defined according to Eq. (17) as

$$(\boldsymbol{\nabla}\mathcal{T}_\epsilon)\times F_{\epsilon n}\hat{\mathbf{n}} - (\boldsymbol{\nabla}\mathcal{T}_\epsilon)\mathcal{T}_\epsilon^{-1}(\partial_n A_{\mu n} + P_e) = \mathbf{J}_e^i,$$

$$-(\boldsymbol{\nabla}\mathcal{T}_\mu)\times A_{\mu n}\hat{\mathbf{n}} - (\boldsymbol{\nabla}\mathcal{T}_\mu)\mathcal{T}_\mu^{-1}(\partial_n F_{\epsilon n} + P_m) = \mathbf{J}_m^i. \tag{24}$$

If conditions (ii) and (iii) are fulfilled the implicit currents are parallel to $\hat{\mathbf{n}}$. This, together with condition (i), ensures the excitation of single-component $\hat{\mathbf{n}}$-directed decoupled vector potentials.

If conditions (ii) and (iii) are violated, i.e., $\boldsymbol{\nabla}\mathcal{T}_\epsilon\times\hat{\mathbf{n}} \neq \mathbf{0}$ and/or $\boldsymbol{\nabla}\mathcal{T}_\mu\times\hat{\mathbf{n}} \neq \mathbf{0}$, then the implicit currents are transversal to the axis $\hat{\mathbf{n}}$ associated with the scalar wave potentials. Like the explicitly defined sources, the implicit current densities can be equivalently reduced to sources parallel to $\hat{\mathbf{n}}$, which are subsequently plugged in the right-hand side of Eq. (23). Thus a solution in terms of the two scalar wave potentials $A_{\mu n}$ and $F_{\epsilon n}$ can be constructed even for a generally nonhomogeneous medium that does not have a distinguished axis. Obviously, in this case $A_{\mu n}$ and $F_{\epsilon n}$ are coupled.

It is now apparent that, regardless of the complexity of the nonhomogeneous medium, the reduction of an EM problem to two scalar wave equations for the pair $(A_{\mu n}, F_{\epsilon n})\hat{\mathbf{n}}$ is possible, and it requires a source scalarization technique which can equivalently transform all explicit and implicit currents into sources parallel to $\hat{\mathbf{n}}$.

III. SCALARIZATION OF SOURCES

The representation of the secondary sources in terms of primary sources via Helmholtz' theorem, Eq. (7), is not unique. Thus, we can define the primary sources in such a way that the transversal currents $(\mathbf{J}^o_{e\tau},\mathbf{J}^o_{m\tau})$ of the original problem are equivalently replaced by a pair of longitudinal potential sources $(G^A_n,G^F_n)\hat{\mathbf{n}}$. Of course, the longitudinal original currents (J^o_{en},J^o_{mn}) do not require transformation. This ensures the excitation of single-component vector potentials $(A_{\mu n},F_{\mu n})\hat{\mathbf{n}}$ in the analyzed volume.

The original transverse currents, $\mathbf{J}^o_{e\tau}$ and $\mathbf{J}^o_{m\tau}$, can be expressed as a superposition of equivalent longitudinal current sources $(\mathbf{J}^e_e=J^e_{en}\hat{\mathbf{n}},\ \mathbf{J}^e_m=J^e_{mn}\hat{\mathbf{n}})$ and equivalent secondary sources, which are given in terms of longitudinal primary sources $\boldsymbol{\mathcal{E}}=\mathcal{E}_n\hat{\mathbf{n}}$, $\boldsymbol{\mathcal{H}}=\mathcal{H}_n\hat{\mathbf{n}}$, as per

$$\mathbf{J}^o_{e\tau}=J^e_{en}\hat{\mathbf{n}}+\nabla\times\mathcal{H}_n\hat{\mathbf{n}}-\nabla P_e,$$

$$\mathbf{J}^o_{m\tau}=J^e_{mn}\hat{\mathbf{n}}-\nabla\times\mathcal{E}_n\hat{\mathbf{n}}-\nabla P_m. \tag{25}$$

Notice that the presence of the equivalent longitudinal current sources $\mathbf{J}^e_e=J^e_{en}\hat{\mathbf{n}}$ and $\mathbf{J}^e_m=J^e_{mn}\hat{\mathbf{n}}$ is required in order to apply the three-dimensional (3D) Helmholtz' representation (7) to the case of transversal currents. For Eq. (25) to hold identically, we must have

$$\mathbf{J}^o_{e\tau}=\nabla_\tau\times\mathcal{H}_n\hat{\mathbf{n}}-\nabla_\tau P_e,$$

$$\mathbf{J}^o_{m\tau}=-\nabla_\tau\times\mathcal{E}_n\hat{\mathbf{n}}-\nabla_\tau P_m, \tag{26}$$

$$J^e_{en}=\partial_n P_e,$$

$$J^e_{mn}=\partial_n P_m. \tag{27}$$

From Eq. (26), the relations between the primary equivalent sources and the transversal original currents are then derived as

$$\nabla^2_\tau\mathcal{H}_n=-(\nabla_\tau\times\mathbf{J}^o_{e\tau})_n,$$

$$\nabla^2_\tau\mathcal{E}_n=(\nabla_\tau\times\mathbf{J}^o_{m\tau})_n, \tag{28}$$

$$\nabla^2_\tau P_e=-\nabla_\tau\cdot\mathbf{J}^o_{e\tau},$$

$$\nabla^2_\tau P_m=-\nabla_\tau\cdot\mathbf{J}^o_{m\tau}. \tag{29}$$

The 2D Poisson equations in Eqs. (28) and (29) are complemented by suitable boundary conditions for the primary sources. The boundary conditions for $\mathcal{E}_n$, $\mathcal{H}_n$, P_e, and P_m at electric and magnetic walls are easily derived from those for the field vectors. They appear as homogeneous Dirichlet or Neumann boundary conditions. Once P_e and P_m are found from Eq. (29), the longitudinal equivalent currents, J^e_{en} and J^e_{mn}, are calculated via Eq. (27). Eqs. (27)–(29) define uniquely the equivalent longitudinal potential sources $(G^A_n,G^F_n)\hat{\mathbf{n}}$ given by

$$G^A_n=J^o_{en}+J^e_{en}+\mathcal{T}_\epsilon\mathcal{E}_n=J^o_{en}+\partial_n P_e+\mathcal{T}_\epsilon\mathcal{E}_n,$$

$$G^F_n=J^o_{mn}+J^e_{mn}+\mathcal{T}_\mu\mathcal{H}_n=J^o_{mn}+\partial_n P_m+\mathcal{T}_\mu\mathcal{H}_n. \tag{30}$$

Once the sources are scalarized, the solution is found in terms of the wave potentials $(A_{\mu n},F_{\epsilon n})$ and their sources (G^A_n,G^F_n). The field can easily be computed from the potentials. Using Eq. (13), we obtain the following expressions for the now scalarized problem

$$E_n=-\mathcal{T}_\mu A_{\mu n}-\partial_n\Phi+\mathcal{E}_n,$$

$$H_n=-\mathcal{T}_\epsilon F_{\epsilon n}-\partial_n\Psi+\mathcal{H}_n,$$

$$\mathbf{E}_\tau=-\nabla_\tau\Phi-\nabla_\tau\times F_{\epsilon n}\hat{\mathbf{n}},$$

$$\mathbf{H}_\tau=-\nabla_\tau\Psi+\nabla_\tau\times A_{\mu n}\hat{\mathbf{n}}, \tag{31}$$

where the scalar potentials Φ and Ψ are calculated according to

$$-\mathcal{T}_\epsilon\Phi=\partial_n A_{\mu n}+P_e,$$

$$-\mathcal{T}_\mu\Psi=\partial_n F_{\epsilon n}+P_m. \tag{32}$$

Equivalently, if Eq. (14) is used, the result is given by

$$\mathcal{T}_\epsilon E_n=-\nabla^2_\tau A_{\mu n}-J^o_{en},$$

$$\mathcal{T}_\mu H_n=-\nabla^2_\tau F_{\epsilon n}-J^o_{mn},$$

$$\mathcal{T}_\epsilon\mathbf{E}_\tau=\nabla_\tau\ \ \partial_n A_{\mu n}-\nabla_\tau\times\ \ \mathcal{T}_\epsilon F_{\epsilon n}\ \ \hat{\mathbf{n}}+\nabla_\tau P_e,$$

$$\mathcal{T}_\mu\mathbf{H}_\tau=\nabla_\tau\ \ \partial_n F_{\epsilon n}+\nabla_\tau\times\ \ \mathcal{T}_\mu A_{\mu n}\hat{\mathbf{n}}+\nabla_\tau P_m. \tag{33}$$

Notice that the secondary equivalent currents $J^e_{mn}=\partial_n P_m$ and $J^e_{en}=\partial_n P_e$ are part of the potential sources and, therefore, they are active (radiating) sources. The portions of the original transversal currents corresponding to $(-\nabla_\tau P_e)$ and $(-\nabla_\tau P_m)$ have been effectively eliminated as nonradiating sources, and they affect the transversal field components only locally as seen in the final two equations of Eq. (33).

It is important to note that the equivalent source transformations are independent of the medium; their defining differential equations (28) and (29) do not contain any of the

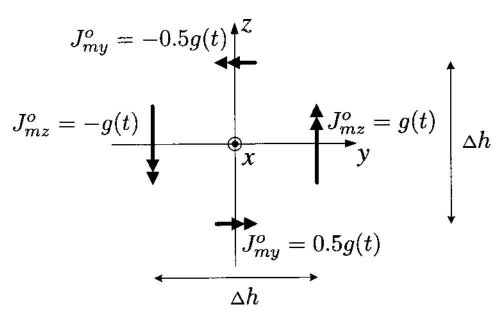

FIG. 1. Distribution of the original magnetic current density in the plane $x=0$.

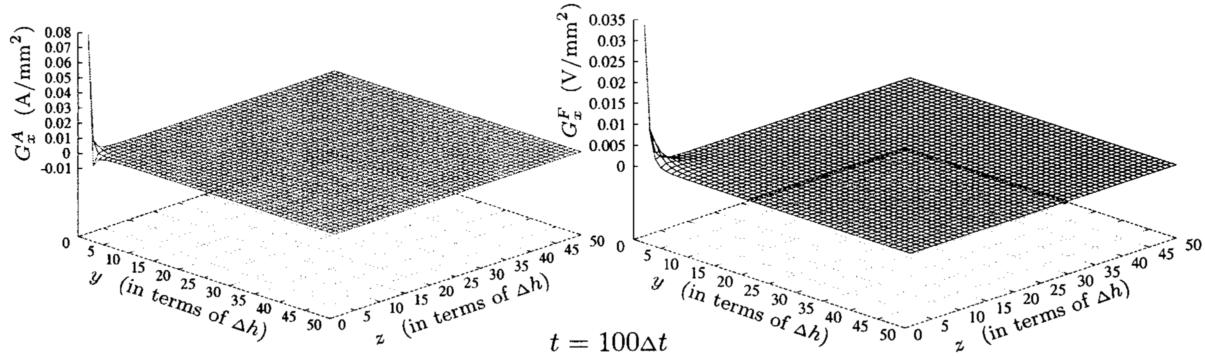

FIG. 2. Planar distribution of the equivalent potential sources (G_x^A, G_x^F) at time $t=100\Delta t$: G_x^A in the plane $x=0$; G_x^F in the plane $x=0.5\Delta h$.

constitutive parameters ϵ, μ, σ_e, and σ_m. The scalarized description obtained here for the case of transient EM problems in an isotropic, nonhomogeneous, lossy medium is in agreement with the frequency-domain analysis of a nonhomogeneous uniaxial medium presented in Refs. [8,16], after appropriate identifications and specializations are carried out. Such an agreement is an important verification since the Hertz potential technique proposed in Refs. [8,16] achieves simultaneous field and source scalarization using a different mathematical approach.

IV. ILLUSTRATIVE NUMERICAL EXAMPLE

The theory of potential sources and source scalarization is illustrated by the EM pulse radiation of an asymmetrical loop of magnetic currents in a plane orthogonal to the distinguished axis, which is chosen as $\hat{\mathbf{n}}=\hat{\mathbf{x}}$. All computations are carried out directly in the time domain using the time-domain wave potential algorithm [9], which treats the wave equations in Eq. (23) with an explicit central finite-difference discretization scheme. The original magnetic currents $\mathbf{J}_{m\tau}^o$ are in the plane $x=0$ such that (see Fig. 1)

$$\nabla_\tau \times \mathbf{J}_{m\tau}^o \neq \mathbf{0}, \quad \nabla_\tau \cdot \mathbf{J}_{m\tau}^o \neq \mathbf{0}. \tag{34}$$

All $\mathbf{J}_{m\tau}^o$ components have the same Gaussian pulse dependence on time:

$$g(t)=\exp[-\alpha(t-t_0)^2], \tag{35}$$

where α and t_0 are numerical constants controlling the width and the position of the pulse along the time axis.

A reference (or original) solution is first obtained by exciting the electric potentials $F_{\epsilon y}$ and $F_{\epsilon z}$ with their respective sources, J_{my}^o and J_{mz}^o. From them, the original field $(\mathbf{E}^o, \mathbf{H}^o)$ is calculated according to Eq. (14). The wave form of each component of the original field is recorded at several locations in order to be compared with the wave form of the respective field component generated by the pair of equivalent potential sources $(G_x^A, G_x^F)\hat{\mathbf{x}}$. The planar distribution of the equivalent potential sources (G_x^A, G_x^F) is computed from (J_{my}^o, J_{mz}^o) using Eqs. (27)–(30). In this particular example, we have $P_e=0$ and $\mathcal{H}_n=0$ because the original set of currents has no electric components. G_x^A and G_x^F are the sources of the wave potentials $(A_{\mu x}, F_{\epsilon x})$ from which the equivalent field $(\mathbf{E}^e, \mathbf{H}^e)$ is calculated using Eq. (33).

A snapshot of the distribution of (G_x^A, G_x^F) in a quadrant of the plane $x=0$ at $t=100\Delta t$ is given in Fig. 2. Here, Δt denotes the discretization step in time. Because of the synchronous behavior in time of all $\mathbf{J}_{m\tau}^o$ components, the

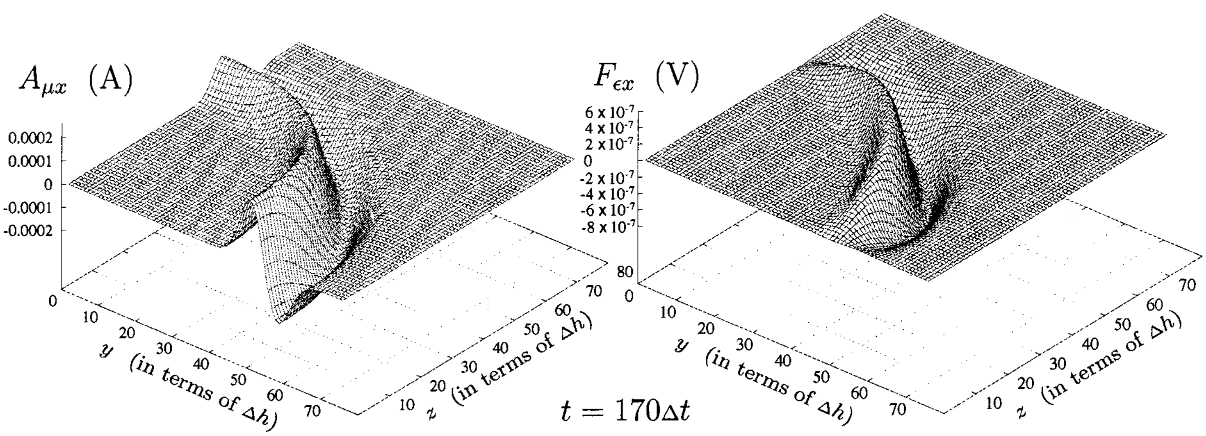

FIG. 3. Distribution of the wave potentials generated by the equivalent potential sources (G_x^A, G_x^F) at time $t=170\Delta t$: $A_{\mu x}$ in the plane $x=0$; $F_{\mu x}$ in the plane $x=0.5\Delta h$.

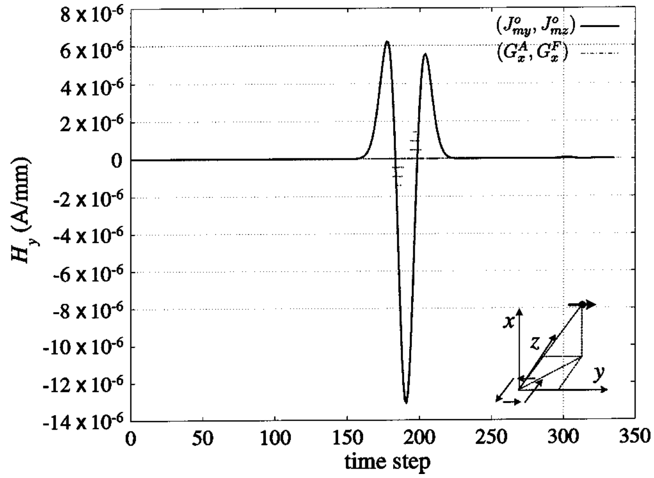

FIG. 4. Comparison of the wave forms of H_y^o and H_y^e generated by the original magnetic loop (J^o_{my},J^o_{mz}), and by the equivalent potential sources (G^A_x,G^F_x).

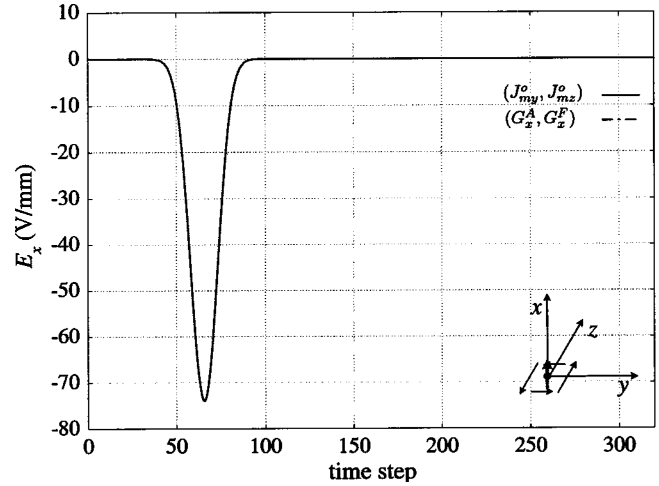

FIG. 5. Comparison of the wave forms of E_x^o and E_x^e generated by the original magnetic loop (J^o_{my},J^o_{mz}), and by the equivalent potential sources (G^A_x,G^F_x).

relative-to-maximum distribution of (G^A_x,G^F_x) in space remains constant in time. The (G^A_x,G^F_x) dependence on time is reflected only by a factor of $g(t)$ applied to the source value at each point in space. A snapshot of the potential pair $(A_{\mu x},F_{\epsilon x})$ distribution in space at $t=170\Delta t$ is shown in Fig. 3. In Figs. 2 and 3, the magnetic potential $A_{\mu x}$ and its sources are plotted in the plane $x=0$, which is the plane of the original magnetic loop. The electric potential $F_{\epsilon x}$ and its sources are calculated at points which are displaced by half a spatial step, $\Delta h/2$, along each axis [9], with respect to the points at which $A_{\mu x}$ is calculated. That is why the $F_{\epsilon x}$ and the G^F_x distributions are plotted in the plane $x=0.5\Delta h$. In this particular example, $F_{\epsilon x}=0$ and $G^F_x=0$ in the plane $x=0$, which is an electric wall.

Figure 4 shows a comparison between the wave forms of the H_y^o component of the original field and the H_y^e component of the equivalent field generated by the equivalent potential sources G^A_x and G^F_x. The observation point is away from the sources, at an elevation angle of $\theta=45°$. The two wave forms are practically indistinguishable from each other. The maximum relative difference $|(H_y^o-H_y^e)/H_y^o|$ is below 10^{-9} when double precision computation is used. It shows excellent agreement bearing in mind the finite-difference nature of the algorithm and the numerical (nonphysical) reflections from the absorbing boundary conditions.

Figure 5 shows a similar comparison only that this time the field component is recorded right inside the source location as shown in the figure inset. Figures 4 and 5 are representative of the behavior of all field components at all observation points. Excellent match of original and equivalent fields is observed everywhere, inside and outside the volume of the original and the equivalent sources.

V. CONCLUSION

The magnetic loop example illustrates very well the concept of equivalent transformation of transversal currents into longitudinal electromagnetic sources obtained in the form of planar distributions. It shows that the field equivalence is preserved everywhere, the volume of the equivalent sources included. The source scalarization technique is in fact a very useful tool when integrated with a computational algorithm, such as the time domain wave potential technique [9]. It allows the solution of problems involving sources of different direction, complex boundary shapes and various types of material nonhomogeneities in terms of only two scalar functions, the scalar wave potentials. This is done through the scalarization of all explicit and implicit EM sources, the latter being induced at material nonhomogeneities and conducting edges. Thus, throughout the computational volume, the analysis is carried out in terms of a single vector potential pair $(A_{\mu n},F_{\epsilon n})\hat{\mathbf{n}}$ of fixed direction $\hat{\mathbf{n}}$.

[1] P.E. Mayes, IRE Trans. Antennas Propag. **6**, 295 (1958).
[2] I.V. Lindell, IEEE Trans. Antennas Propag. **36**, 1382 (1988).
[3] I.V. Lindell, *Methods of Electromagnetic Field Analysis* (Clarendon Press, Oxford, 1992), p. 192.
[4] R.F. Harrington, *Time–Harmonic Electromagnetic Fields*, Classic Textbook Reissue, Vol. 131 (McGraw-Hill, New York, 1961).
[5] D.S. Jones, *Acoustic and Electromagnetic Waves* (Oxford University Press, New York, 1986), p. 27.
[6] D.R. Wilton, IEEE Trans. Antennas Propag. **28**, 111 (1980).
[7] A. Nisbet, Proc. R. Soc. London, Ser. A **240**, 375 (1957).
[8] W.S. Weiglhofer, Int. J. Appl. Electromagn. Mech. **11**, 131 (2000).
[9] N.K. Georgieva, IEEE Trans. Microwave Theory Tech. **50**, 1950 (2002).
[10] N.K. Georgieva and W.S. Weiglhofer, report, 2002 (unpub-

lished).
[11] W.S. Weiglhofer and N.K. Georgieva, Electromagnetics (to be published).
[12] W.S. Weiglhofer and A. Lakhtakia, Arch. Elektron. Üebertr. **50**, 389 (1996).
[13] A. Taflove, *Computational Electrodynamics: The Finite-Difference Time-Domain Method* (Artech House, Boston, MA, 1995).
[14] J. Schwinger, L.L. DeRaad, Jr., K.A. Milton, and W.-Y. Tsai, *Classical Electrodynamics* (Perseus Books, Reading, MA, 1998).
[15] A.J. Devaney and E. Wolf, Phys. Rev. D **8**, 1044 (1973).
[16] W.S. Weiglhofer, in *Electromagnetic Fields in Unconventional Materials and Structures*, edited by O.N. Singh and A. Lakhtakia (Wiley, New York, 2000), p. 1.